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Discrete derivatives and symmetries of difference equations

D Levi[1], **J Negro**[2] **and M A del Olmo**[2]

[1] Departimento di Fisica, Universitá Roma Tre and INFN–Sezione di Roma Tre, Via della Vasca Navale 84, 00146 Roma, Italy
[2] Departamento de Física Teórica, Universidad de Valladolid, E-47011, Valladolid, Spain

E-mail: levi@amaldi.fis.uniroma3.it, jnegro@fta.uva.es and olmo@fta.uva.es

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Abstract

We show with an example of the discrete heat equation that for any given discrete derivative we can construct a nontrivial Leibniz rule suitable for finding the symmetries of discrete equations. In this way we obtain a symmetry Lie algebra, defined in terms of shift operators, isomorphic to that of the continuous heat equation.

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1. Introduction

Lie point symmetries were introduced by Sophus Lie for solving differential equations. They turn out to provide one of the most efficient methods for obtaining exact analytical solutions of partial differential equations [1–3]. This essentially continuous method has been recently extended to the case of discrete equations [4–6].

Let us write a general difference equation, involving, for notational simplicity, one scalar function $u(x)$ of p independent variables $x = (x_1, x_2, \dots, x_p)$ evaluated at a finite number of points on a lattice. Symbolically we write

$$E(x, T^a u(x),\ T^{b_i} \Delta_{x_i} u(x), T^{c_{ij}} \Delta_{x_i} \Delta_{x_j} u(x), \dots) = 0 \tag{1.1}$$

where E is some given function of its arguments,

$$T^a u(x) := \{T_{x_1}^{a_1} T_{x_2}^{a_2} \cdots T_{x_p}^{a_p}\, u(x)\}_{a_i=m_i}^{n_i} \qquad a = (a_1, a_2, \dots, a_p) \quad i = 1, 2, \dots, p$$

with a_i, m_i, n_i fixed integers ($m_i \leqslant n_i$), and

$$T_{x_i}^{a_i} u(x) = u(x_1, x_2, \dots, x_{i-1}, x_i + a_i\sigma_i, x_{i+1}, \dots, x_p)$$

(the other shift operators T^{b_i}, $T^{c_{ij}}$ are defined in a similar way), Δ_{x_i} is a difference operator which in the continuous limit goes into the derivative and σ_i is the positive lattice spacing in the uniform lattice of the variable x_i ($i = 1, \dots, p$).

The simplest extension of the continuous case is when the symmetry transformation which leaves the equation on the lattice invariant depends just on $u(x)$ and not on its shifted values,

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which used to be called intrinsic point transformations [4]. In this case the whole theory succeeds but the resulting transformations are somehow trivial and often provide not very interesting solutions.

In [7] it has been proved that the intrinsic transformations can be extended for linear equations by considering symmetries where the transformed function $\tilde{u}(x)$ depends not only on the old $u(x)$ and x but also on the function u in shifted points $T^{a_i}u(x)$ on the lattice.

The results presented in [7] were obtained by considering the difference operator

$$\Delta_{x_i} \equiv \Delta_{x_i}^{+} = \frac{T_{x_i} - 1}{\sigma_i}. \tag{1.2}$$

This difference operator is the simplest one which, when $\sigma_i \to 0$, goes into the standard derivative with respect to x_i.

Obviously, more general definitions of the difference operator can be introduced and one would like to be able to prove that the obtained symmetries are independent of the discretization of the difference operator one is considering. Among the possibilities, let us mention

$$\Delta_{x_i}^{-} = \frac{1 - T_{x_i}^{-1}}{\sigma_i} \tag{1.3}$$

corresponding to the left derivative, and the symmetric derivative (which goes into the derivative with respect to x_i up to terms of order σ_i^2)

$$\Delta_{x_i}^{\mathrm{s}} = \frac{T_{x_i} - T_{x_i}^{-1}}{2\sigma_i}. \tag{1.4}$$

Using the techniques of approximate differentiation [8] one could write more complicated formulas for difference operators which, however, are out of the scope of this paper.

In the following we will show that by a nontrivial definition of the Leibniz rule we can construct symmetries for any difference operator. Section 2 is devoted to reviewing the Lie group formalism for difference equations. In section 3 we apply it to the case of the discrete heat equation using the discrete derivatives (1.3), (1.4), finding difficulties in the case (1.4). In section 4 we show via examples how one can introduce in a consistent way the Leibniz rule and use this result to obtain determining equations equivalent to those for the standard discrete derivatives (i.e. $\Delta_{x_i}^{\pm}$). This fact shows that we obtain different representations of the same symmetry group. In the conclusions we outline how to write the determining equations when one uses a generic difference operator by using the corresponding Leibniz rule obtained with our approach.

2. Lie symmetries of difference equations

Among the different algebraic methods for calculating the symmetries of discrete equations [5–10] we will make use in this paper of the approach presented in [10], based on the formalism of evolutionary vector fields for differential equations [1].

For a difference equation of order N like that given in (1.1) the infinitesimal symmetry vectors in evolutionary form, which in the continuous limit go over to point symmetries, take the general expression

$$X_e \equiv Q\partial_u = \left(\sum_i \xi_i(x, T^a u, \sigma_x, \sigma_t) T^b \Delta_{x_i} u - \phi(x, T^c u, \sigma_x, \sigma_t)\right)\partial_u \tag{2.1}$$

where $\xi_i(x, T^a u, \sigma_x, \sigma_t)$ and $\phi(x, T^c u, \sigma_x, \sigma_t)$ are functions which in the continuous limit go over to $\xi_i(x, u)$ and $\phi(x, u)$, respectively.

The vector fields X_e generate the symmetry group of the discrete equation (1.1), whose elements transform solutions $u(x)$ of the equation into solutions $\tilde{u}(x)$. Since equation (1.1) is of order N in the difference operators, the Nth prolongation of X_e must verify the invariance condition

$$pr^N X_e E|_{E=0} = 0. \tag{2.2}$$

The group generated by the prolongations also transforms solutions into solutions, and $\Delta_{x_i} u$, $\Delta_{x_i}\Delta_{x_j} u, \ldots$ (up to order N) into the variations of $\tilde{u}$ with respect to x_i. The formula of $pr^N X_e$ is given by

$$pr^N X_e = \sum_a T^a Q \partial_{T^a u} + \sum_{b_i} T^{b_i} Q^{x_i} \partial_{T^{b_i}\Delta_{x_i} u} + \sum_{c_{ij}} T^{c_{ij}} Q^{x_i x_j} \partial_{T^{c_{ij}}\Delta_{x_i}\Delta_{x_j} u} + \cdots. \tag{2.3}$$

The summations in (2.3) are over all the sites present in (1.1), and Q^{x_i}, $Q^{x_i x_j}, \ldots$ are total variations of Q, i.e.,

$$Q^{x_i} = \Delta_{x_i}^T Q \qquad Q^{x_i x_j} = \Delta_{x_i}^T \Delta_{x_j}^T Q, \ldots$$

where the partial variation Δ_{x_i} is defined by

$$\begin{aligned}\Delta_x f(x, u(x), \Delta_x u(x), \ldots) = {} & \frac{1}{\sigma}[f(x+\sigma, u(x), (\Delta_x u)(x), \ldots) \\ & - f(x, u(x), \Delta_x u(x), \ldots)] \qquad x = x_i\end{aligned}$$

and the total variation $\Delta_{x_i}^T$ by

$$\begin{aligned}\Delta_x^T f(x, u(x), \Delta_x u(x), \ldots) = {} & \frac{1}{\sigma}[f(x+\sigma, u(x+\sigma), (\Delta_x u)(x+\sigma), \ldots) \\ & - f(x, u(x), \Delta_x u(x), \ldots)] \qquad x = x_i.\end{aligned}$$

Note that expressions (2.1)–(2.3) are analogous to those of the continuous case [1] and can be derived in a similar way [10].

The symmetries of equation (1.1) are given by condition (2.2), which give rise to determining equations for ξ_i and ϕ as the coefficients of linearly independent expressions in the discrete derivatives $T^a \Delta_{x_i} u$, $T^b \Delta_{x_i x_j} u, \ldots$.

The Lie commutators of the vector fields X_e are obtained by commuting their first prolongations and projecting onto the symmetry algebra $\mathcal{G}$, i.e.,

$$\begin{aligned}[X_{e1}, X_{e2}] &= [pr^1 X_{e1}, pr^1 X_{e2}]|_{\mathcal{G}} \\ &= \left(Q_1 \frac{\partial Q_2}{\partial u} - Q_2 \frac{\partial Q_1}{\partial u} + Q_1^{x_i} \frac{\partial Q_2}{\partial u_{x_i}} - Q_2^{x_i} \frac{\partial Q_1}{\partial u_{x_i}}\right) \partial_u\end{aligned} \tag{2.4}$$

where the ∂u_{x_i} terms disappear after projection onto $\mathcal{G}$.

The formalism presented above may become quite involved and if the system is nonlinear it is almost impossible to obtain a result since the number of terms to consider is *a priori* infinite. The situation is simpler for linear equations where we can use a reduced ansatz. Thus, in this case we assume that the evolutionary vectors (2.1) have the form

$$X_e = \left(\sum_i \xi_i(x, T^a, \sigma_x, \sigma_t)\Delta_{x_i} u - \phi(x, T^a, \sigma_x, \sigma_t) u\right)\partial_u. \tag{2.5}$$

Now the vector fields X_e can be written as $X_e = (\hat{X} u)\partial_u$, i.e.,

$$\hat{X} = \sum_i \xi_i(x, T^a, \sigma_x, \sigma_t)\Delta_{x_i} - \phi(x, T^a, \sigma_x, \sigma_t). \tag{2.6}$$

The operators $\hat{X}$ given by the above ansatz may span only a subalgebra of the whole Lie symmetry algebra (see [10]).

3. Discrete heat equation

Let us consider the equation

$$(\Delta_t - \Delta_{xx})u(x) = 0 \tag{3.1}$$

which is a discretization of the heat equation. As the equation is linear we can consider an evolutionary vector field of the form

$$X_e \equiv Q\partial_u = (\tau\Delta_t + \xi\Delta_x u + fu)\partial_u \tag{3.2}$$

where τ, ξ and f are arbitrary functions of $x, t, T_x\ T_t$, σ_x and σ_t. As T_x and T_t are operators not commuting with x and t, respectively, τ, ξ and f are operator valued functions. Since equation (3.1) is a second-order difference equation it is necessary to use the second prolongation. The determining equation is

$$\Delta_t^T Q - \Delta_{xx}^T Q|_{\Delta_{xx}u=\Delta_t u} = 0 \tag{3.3}$$

which explicitly reads

$$\Delta_t(\xi\Delta_x u) + \Delta_t(\tau\Delta_t u) + \Delta_t(fu) - [\Delta_{xx}(\xi\Delta_x u) + \Delta_{xx}(\tau\Delta_t u) + \Delta_{xx}(fu)]|_{\Delta_{xx}u=\Delta_t u} = 0. \tag{3.4}$$

We had no need to give the explicit form of the operator Δ to obtain equation (3.4). Only when developing expression (3.4) do we need to apply the Leibniz rule and, hence, the results will depend on the given definition of the discrete derivative.

3.1. *Symmetries in the right (left) discrete derivative case*

Choosing as in [7, 10] the derivative Δ^+ and consequently the Leibniz rule

$$\Delta^+(fg) = \Delta^+(f)Tg + f\Delta^+g \tag{3.5}$$

we obtain the following set of determining equations

$$\begin{aligned}
&\Delta_x^+\tau = 0\\
&(\Delta_t^+\tau)T_t - 2(\Delta_x^+\xi)T_x = 0\\
&(\Delta_t^+\xi)T_t - (\Delta_{xx}^+\xi)T_x^2 - 2(\Delta_x^+ f)T_x = 0\\
&(\Delta_t^+ f)T_t - (\Delta_{xx}^+ f)T_x^2 = 0
\end{aligned} \tag{3.6}$$

by equating to zero the coefficients of $\Delta_{xt}u$, $\Delta_t u$, $\Delta_x u$ and u, respectively. The solution of (3.6) gives

$$\begin{aligned}
\tau &= t^{(2)}\tau_2 + t\tau_1 + \tau_0\\
\xi &= \tfrac{1}{2}x(\tau_1 + 2t\tau_2)T_t T_x^{-1} + t\xi_1 + \xi_0\\
f &= \tfrac{1}{4}x^{(2)}\tau_2 T_t^2 T_x^{-2} + \tfrac{1}{2}t\tau_2 T_t + \tfrac{1}{2}x\xi_1 T_t T_x^{-1} + \gamma
\end{aligned} \tag{3.7}$$

where τ_0, τ_1, τ_2, ξ_0, ξ_1 and γ are arbitrary functions of T_x, T_t and of the spacings σ_x and σ_t, and $x^{(n)}$, $t^{(n)}$ are the Pochhammer symbols given by, for instance,

$$x^{(n)} = x(x - \sigma_x)\cdots(x - (n-1)\sigma_x).$$

By a suitable choice of the functions τ_i, ξ_i, and γ we obtain the following symmetries:

$$\begin{aligned}
&P_0 = (\Delta_t u)\partial_u \qquad (\tau_0 = 1)\\
&P_1 = (\Delta_x u)\partial_u \qquad (\xi_0 = 1)\\
&W = u\partial_u \qquad (\gamma = 1)\\
&B = (2tT_t^{-1}\Delta_x u + xT_x^{-1}u)\partial_u \qquad (\xi_1 = 2T_t^{-1})\\
&D = (2tT_t^{-1}\Delta_t u + xT_x^{-1}\Delta_x u + \tfrac{1}{2}u)\partial_u \qquad (\tau_1 = 2T_t^{-1}, \gamma = \tfrac{1}{2})\\
&K = (t^2T_t^{-2}\Delta_t u - \sigma_t tT_t^{-2}\Delta_t u + txT_t^{-1}T_x^{-1}\Delta_x u\\
&\qquad + \tfrac{1}{4}x^2T_x^{-2}u - \tfrac{1}{4}\sigma_x xT_x^{-2}u + \tfrac{1}{2}tT_t^{-1}u)\partial_u \qquad (\tau_2 = T_t^{-2})
\end{aligned} \tag{3.8}$$

that close into a six-dimensional Lie algebra which is isomorphic to the symmetry algebra of the continuous heat equation. In [7] another realization of this algebra was obtained by a different procedure and used to find symmetric solutions of the discrete heat equation.

A second choice for the discrete derivative is Δ^- and now the Leibniz rule becomes

$$\Delta^-(fg) = \Delta^-(f)T^{-1}g + f\Delta^- g. \tag{3.9}$$

This gives the same results (3.7) and (3.8) provided we make the substitution $T \to T^{-1}$.

3.2. Symmetries in the symmetric discrete derivative case

Next let us consider the case of the symmetric derivative Δ^{s}. It seems natural to propose the following Leibniz rule

$$\Delta^{\mathrm{s}}(fg) = \Delta^{\mathrm{s}}(f)T^{-1}g + (Tf)\Delta^{\mathrm{s}}g. \tag{3.10}$$

In this case taking as before coefficients of the different discrete derivatives of u ($\Delta_{tt}u$, $\Delta_{xt}u$, $\Delta_t u$, $\Delta_x u$ and u, respectively), the determining equations are

$$\begin{aligned}
&T_t\tau - T_x^2\tau = 0\\
&T_t\xi - T_x^2\xi - 2(T_x\Delta_x^{\mathrm{s}}\tau)T_x^{-1} = 0\\
&(\Delta_t^{\mathrm{s}}\tau)T_t^{-1} + (T_t f) - 2(T_x\Delta_x^{\mathrm{s}}\xi)T_x^{-1} - (\Delta_{xx}^{\mathrm{s}}\tau)T_x^{-2} - T_x^2 f = 0\\
&(\Delta_t^{\mathrm{s}}\xi)T_t^{-1} - (\Delta_{xx}^{\mathrm{s}}\xi)T_x^{-2} - 2(T_x\Delta_x^{\mathrm{s}} f)T_x^{-1} = 0\\
&(\Delta_t^{\mathrm{s}} f)T_t^{-1} - (\Delta_{xx}^{\mathrm{s}} f)T_x^{-2} = 0.
\end{aligned} \tag{3.11}$$

Note that there is one equation (the first one, associated to $\Delta_{tt}u$) more than in the two previous cases. This implies that the solution of such equations (3.11) is just

$$\tau = \tau_0 \qquad \xi = \xi_0 \qquad f = f_0$$

where τ_0, ξ_0, and f_0 are arbitrary functions of T_x, T_t and of the spacings σ_x and σ_t.

Obviously, in this last case our naive approach does not allows us to recover the whole symmetry algebra of the heat equation (3.8). In fact, we will show in next section that the decomposition of equation (3.4) into equations (3.11) via Leibniz rule (3.10) is not the most appropriate one.

4. Leibniz rule and symmetries with symmetric derivatives

Let us consider a formal way to obtain the Leibniz rule in the continuous case which is easily extendible to the case of discrete derivatives. This result will lead us to the symmetries of a class of discrete equations independent of the discrete derivative used.

Starting from the well known result $[\partial_x, x] = 1$, by algebraic methods we find that $[\partial_x, f(x)] = f'(x)$ (at least for analytic functions). Consequently, the Leibniz rule $\partial_x(fg) = fg' + f'g$ can be derived.

Now, let us consider the commutator with the differential operator substituted by a difference one. By direct computation we have

$$[\Delta_x^{\pm}, x] = T_x^{\pm 1} \qquad [\Delta_x^{\mathrm{s}}, x] = \frac{T_x + T_x^{-1}}{2}. \tag{4.1}$$

By introducing a function $\beta_x = \beta(T_x)$ we can always rewrite the commutation relations (4.1) as

$$[\Delta_x, x\beta_x] = 1. \tag{4.2}$$

Let us note that $\Delta_x\beta_x = \beta_x\Delta_x$ and $\Delta_t\beta_x = \beta_x\Delta_t$.

For the standard left and right derivatives $\Delta_x^{\pm 1}$ we easily find that $\beta_x^{\pm} = T_x^{\mp 1}$, and we find that

$$[\Delta_x^{\pm}, f(x)\beta^{\pm}] = (\Delta_x^{\pm} f(x)).$$

From this last expression we recover Leibniz rules (3.5) and (3.9).

For the symmetric discrete derivative Δ_x^{s} we obtain the formal expression

$$\beta_x^{\mathrm{s}} = 2(T_x + T_x^{-1})^{-1}$$

and

$$[\Delta_x^{\mathrm{s}}, f(x)\beta_x^{\mathrm{s}}] = (\Delta_x^{\mathrm{s}} f(x))T_x\beta_x^{\mathrm{s}} + (T_x^{-1} f(x) - f(x))\beta_x^{\mathrm{s}}\Delta_x^{\mathrm{s}}. \tag{4.3}$$

From relation (4.3) we obtain the Leibniz rule

$$\Delta_x^{\mathrm{s}}(f(x)g(x)) = f(x)\Delta_x^{\mathrm{s}} g(x) + \left[\frac{1}{\sigma_x}((T_x^{-1} - 1)f(x))(T_x - (\beta_x^{\mathrm{s}})^{-1}) + (\Delta_x^{\mathrm{s}} f(x))T_x\right]g(x). \tag{4.4}$$

Let us note that expression (4.4) looks very different to (3.10).

Formula (4.4) allows us to write down the determining equations (3.4) as

$$\begin{aligned}
&((1 - T_x^{-1})\tau)T_x^{-1} + ((T_x - 1)\tau)T_x = 0\\
&\frac{1}{2\sigma_t}[((1 - T_t^{-1})\tau)T_t^{-1} + ((T_t - 1)\tau)T_t] - \frac{1}{\sigma_x}[((1 - T_x^{-1})\xi)T_x^{-1} + ((T_x - 1)\xi)T_x] = 0\\
&\frac{1}{2\sigma_t}[((1 - T_t^{-1})\xi)T_t^{-1} + ((T_t - 1)\xi)T_t] - \frac{1}{\sigma_x}[((1 - T_x^{-1})f)T_x^{-1} + ((T_x - 1)f)T_x]\\
&\qquad - \frac{1}{4\sigma_x^2}[((1 - T_x^{-1})^2\xi)T_x^{-2} + 2((T_x + T_x^{-1} - 2)\xi) + ((T_x - 1)^2\xi)T_x^2] = 0\\
&\frac{1}{2\sigma_t}[((1 - T_t^{-1})f)T_t^{-1} + ((T_t - 1)f)T_t] - \frac{1}{4\sigma_x^2}[((1 - T_x^{-1})^2 f)T_x^{-2}\\
&\qquad +2((T_x + T_x^{-1} - 2)f) + ((T_x - 1)^2 f)T_x^2] = 0
\end{aligned} \tag{4.5}$$

obtained as coefficients of $\Delta_{xt}u$, $\Delta_t u$, $\Delta_x u$ and u, respectively. Thus, we obtain the same number of equations as in the cases of the standard discrete derivatives $\Delta^{\pm}$.

The solution of equations (4.5) is given by

$$\begin{aligned}
\tau^{\mathrm{s}} &= t^{(2)}\tau_2 + t\tau_1 + \tau_0\\
\xi^{\mathrm{s}} &= \tfrac{1}{2}x(2t\tau_2 + \tau_1 + \sigma_t T_t^{-1}\beta_t^{\mathrm{s}}\tau_2)(\beta_t^{\mathrm{s}})^{-1}\beta_x^{\mathrm{s}} + t\xi_1 + \xi_0\\
f^{\mathrm{s}} &= \tfrac{1}{4}x^{(2)}\tau_2(\beta_x^{\mathrm{s}})^2(\beta_t^{\mathrm{s}})^{-2} + \tfrac{1}{2}x\xi_1\beta_x^{\mathrm{s}}(\beta_t^{\mathrm{s}})^{-1} + \tfrac{1}{4}x\sigma_x\tau_2 T_x^{-1}(\beta_x^{\mathrm{s}})^3(\beta_t^{\mathrm{s}})^{-2} + \tfrac{1}{2}t\tau_2(\beta_t^{\mathrm{s}})^{-1} + f_0
\end{aligned} \tag{4.6}$$

where τ_2, τ_1, τ_0, ξ_1, ξ_0 and f_0 are arbitrary functions of T_x, T_t, σ_x and σ_t.

From (4.6) and (3.2) we obtain, with a suitable choice of τ_2, τ_1, τ_0, ξ_1, ξ_0 and f_0, the following symmetries

$$\begin{aligned}
&P_0^{\mathrm{s}} = (\Delta_t^{\mathrm{s}} u)\partial_u \qquad (\tau_0 = 1)\\
&P_1^{\mathrm{s}} = (\Delta_x^{\mathrm{s}} u)\partial_u \qquad (\xi_0 = 1)\\
&W^{\mathrm{s}} = u\partial_u \qquad (f_0 = 1)\\
&B^{\mathrm{s}} = (2t\beta_t^{\mathrm{s}}\Delta_x^{\mathrm{s}} u + x\beta_x^{\mathrm{s}} u)\partial_u \qquad (\xi_1 = 2\beta_t^{\mathrm{s}})\\
&D^{\mathrm{s}} = (2t\beta_t^{\mathrm{s}}\Delta_t^{\mathrm{s}} u + x\beta_x^{\mathrm{s}}\Delta_x^{\mathrm{s}} u + \tfrac{1}{2}u)\partial_u \qquad (\tau_1 = 2\beta_t^{\mathrm{s}},\ f_0 = \tfrac{1}{2})\\
&K^{\mathrm{s}} = ((t^2(\beta_t^{\mathrm{s}})^2 - t{\sigma_t}^2(\beta_t^{\mathrm{s}})^3\Delta_t^{\mathrm{s}})\Delta_t^{\mathrm{s}} u + tx\beta_t^{\mathrm{s}}\beta_x^{\mathrm{s}}\Delta_x^{\mathrm{s}} u \qquad (\tau_2 = \beta_t^{\mathrm{s}2},\\
&\qquad - \tfrac{1}{4}x{\sigma_x}^2(\beta_x^{\mathrm{s}})^3\Delta_x^{\mathrm{s}} u + \tfrac{1}{4}x^2(\beta_x^{\mathrm{s}})^2 u + \tfrac{1}{2}t\beta_t^{\mathrm{s}} u)\partial_u \qquad \tau_1 = \sigma_t\beta_t^{\mathrm{s}2} - \sigma_t^2\beta_t^{\mathrm{s}3}\Delta_t^{\mathrm{s}}).
\end{aligned} \tag{4.7}$$

They close the same six-dimensional Lie algebra generated by the operators (3.8). The above result deserves some comments. First, there appear functions β_t^{s} (β_x^{s}) of T_t (T_x) that can only be understood as infinite series developments. Therefore, some of the above symmetries (4.7) do not have a local character in the sense that they are not polynomials in $T_t^{\pm 1}$, $T_x^{\pm 1}$. Although such symmetries give rise to the classical symmetries in the limit $\sigma_x \to 0$, $\sigma_t \to 0$, one of them (K) also includes, surprisingly, a term in $(\Delta_t)^2$ (which vanishes in the continuous limit since it has a factor σ_t^2).

This symmetry algebra can be applied to construct bases of symmetric solutions as well as the related special functions following the method used in [7] and [11].

5. Conclusions

Analysing the different Leibniz rules used in sections 3 and 4, we see that when we obtain the correct result the Leibniz rule must have the form

$$\Delta_x(f(x)g(x)) = f(x)\Delta_x g(x) + D_x(f(x))g(x) \tag{5.1}$$

where D_x is a function of T_x, β_x and σ_x (we should have written $D_x(f(x); T_x, \beta_x, \sigma_x)$, with β_x given by equation (4.2), but for the sake of brevity we will simply write $D_x(f)$, and similarly to obtain $D_t(f)$). Once we have chosen a particular discrete derivative Δ_x we can write the explicit expression of D_x. In particular, for $\Delta_x^\pm$ and Δ_x^{s} their corresponding $D_x(f)$ functions are

$$\begin{aligned} D_x^\pm(f) &= (\Delta_x^\pm(f))(\beta_x^\pm)^{-1} = (\Delta_x^\pm(f))T_x^{\pm 1} \\ D_x^{\mathrm{s}}(f) &= \frac{1}{\sigma_x}((T_x^{-1}-1)f)(T_x - (\beta_x^{\mathrm{s}})^{-1}) + (\Delta_x^{\mathrm{s}} f)T_x \\ &= ((T_x^{-1}-1)f)\Delta_x^{\mathrm{s}} + ((\Delta_x^{\mathrm{s}})f)T_x] \\ &= \frac{1}{2\sigma_x}[((1-T_x^{-1})f)T_x^{-1} + ((T_x-1)f)T_x]. \end{aligned} \tag{5.2}$$

Using the general Leibniz rule (5.1) for an arbitrary discrete derivative we obtain from (3.4), equating to zero the coefficients of $\Delta_{xt}u$, $\Delta_t u$, $\Delta_x u$ and u, respectively, the following set of determining equations

$$\begin{aligned} &D_x(\tau) = 0 \\ &D_t(\tau) - 2D_x(\xi) = 0 \\ &D_t(\xi) - D_{xx}(\xi) - 2D_x(f) = 0 \\ &D_t(f) - D_{xx}(f) = 0 \end{aligned} \tag{5.3}$$

where $D_{xx}(f) = D_x(D_x(f))$.

For the cases $\Delta^\pm$ and Δ^{s} from (5.3) we recover the determining equations (3.6) and (4.5), respectively.

We always have four equations, whose general solution will depend on the formal expression of the corresponding Δ. In fact, as long as we are able to find for the discrete derivative Δ the operator β satisfying the commutator (4.2) this will allow us to give particular solutions to the determining equations (5.3) keeping the same Lie structure of the symmetries of the classical heat equation. Of course, some of these discrete symmetries can have a rather complicated expression with a nonlocal character, but this feature is a consequence of the structure of the operator β.

This procedure can be straightforwardly applied to other discretizations such as the wave equation [11] or even equations including a potential term provided that for the corresponding discrete derivative, Δ, there exists a β such that relation (4.2) is fulfilled. In general we can state that if the equation $E(x, \Delta) = 0$ can be rewritten as $E'(x\beta, \Delta) = 0$, then the algebra of its discrete symmetries will include a subalgebra isomorphic to the Lie point symmetry algebra of the continuous equation $E'(x, \partial_x) = 0$ (for further details see [12]).

Nonlinear equations need additional improvements in order to have reasonable determining equations.

Something similar happens with discrete equations on a q-lattice: once we have a solution for β we can derive symmetry operators with a Lie structure (a situation which is seldom considered). On the other hand, if we are more interested in local symmetries, with some restrictions, the natural structure is that of a q-algebra. However, these issues are out of our present scope and they will be published elsewhere.

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